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SPECIAL CIRCLES IN MECHANICS

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Abstract—W. Thomson and P. G. Tait (*Principles of Mechanics and Dynamics*, Dover, 1962) noted that for any deformation at any particle there is at least one plane circle of material elements all of which suffer precisely the same stretch. It has been shown that these material elements suffer no shear. These are typical of the results which may be obtained by considering ellipsoids associated with real non-singular second-order tensors. It is shown that the results may also be obtained provided the second-order tensor has at least one zero eigenvalue. Applications to strain, both finite and infinitesimal, rate of strain and stress are presented.

1. INTRODUCTION

This paper is concerned with real second-order tensors which occur in mechanics, such as strain and stress tensors. An ellipsoid may be associated with any non-singular second-order tensor. Occasionally, if, for example a tensor such as the rate of strain tensor is singular, it may nevertheless have an ellipsoid associated with it for the purpose of obtaining certain properties. Generally an ellipsoid has two central circular sections—of course in the case of a spheroid (ellipsoid of revolution) there is only one central circular section whilst in the case of a sphere there is an infinity of such sections. It is of value to consider these central circular sections for the various ellipsoids because generally they have special identifiable properties. Such a special property was first noted by Thomson and Tait (1962). They observed for a general deformation at any particle X, that all material elements which undergo the same extension lie on a cone except in one particular case where the cone becomes two planes. These two planes are the planes of the central circular section of the strain ellipsoid $d\mathbf{X}^T\mathbf{C}\,d\mathbf{X} = 1$ where $\mathbf{C}$ is the left Cauchy–Green strain tensor. It may also be shown (Hayes, 1988) that material elements in these planes suffer no shear—the angle θ (say) between a pair of material elements before deformation is also the angle between the stretched elements after the deformation.

Similar results are valid for infinitesimal strain and for rate of strain. Also we present results for area elements and for stress tensors.

First we recall a general result for non-singular second-order tensors and prove that the same general result is also valid when the tensor possesses only one zero eigenvalue.

Notation

The summation convention is used throughout—repeated suffixes imply summation over 1, 2, 3. The vectors $\mathbf{n}$, $\mathbf{m}$, $\mathbf{N}$ and $\mathbf{M}$ are unit vectors throughout.

2. BASIC RESULTS

Here we provide the mathematical setting for the results which follow. We recall the general theorem (Hayes, 1990) for non-singular second-order tensors, and derive an identity which is very useful in visualising the role of the central circular sections. Then we partially extend the theorem to include singular tensors with one zero eigenvalue.

Theorem. Let $\boldsymbol{\Psi}$ be any real non-singular second-order tensor. Let $\mathbf{N}$ be an arbitrary unit vector and let the vector $\mathbf{R}_{(\mathbf{N})}$ be given by $\mathbf{R}_{(\mathbf{N})} = \boldsymbol{\psi}\mathbf{N}$. Then there exists at least one circle of special directions for $\mathbf{N}$ such that the corresponding $\mathbf{R}_{(\mathbf{N})}$ also lie on a circle. Further, if θ is the angle between any pair of special directions, say $\mathbf{N}'$ and $\mathbf{N}''$, then θ is also the angle between $\mathbf{R}_{(\mathbf{N}')}$ and $\mathbf{R}_{(\mathbf{N}'')}$. The number of circles of special directions is equal to the number of central circular sections of the ellipsoid $\mathbf{X}^T\boldsymbol{\psi}^T\boldsymbol{\psi}\mathbf{X} = 1$: one, if the ellipsoid is a spheroid ; two, if the axes of the ellipsoid are all unequal ; infinity, if the ellipsoid is a sphere.

Because some of the ideas are needed in the sequel we present the general proof here, referring to the paper (Hayes, 1990) for the cases of equal eigenvalues.

Proof. We have

$$R_i = \psi_{iA}N_A, \tag{1}$$

where repeated suffixes are summed over 1, 2, 3. Let $\boldsymbol{\phi}$ be defined by

$$\boldsymbol{\phi} = \boldsymbol{\psi}^T\boldsymbol{\psi}, \quad \phi_{AB} = \psi_{iA} \times \psi_{iB}. \tag{2}$$

Then $\boldsymbol{\phi}$ is real, symmetric and positive definite. It possesses real positive eigenvalues ϕ_α (say). We assume, for the moment, that these are ordered $\phi_1 > \phi_2 > \phi_3$, and let $\mathbf{J}^{(\alpha)}$ be the corresponding mutually orthogonal unit eigenvectors of $\boldsymbol{\phi}$. Then

$$\phi_{AB} = \phi_2\delta_{AB} + \{\tfrac{1}{2}(\phi_1 - \phi_3)\}(L_AM_B + L_BM_A), \tag{3}$$

where the unit vectors $\mathbf{L}$ and $\mathbf{M}$ are given by

$$\begin{aligned} \sqrt{(\phi_1-\phi_3)}\mathbf{L} &= \sqrt{(\phi_1-\phi_2)}\mathbf{J}^{(1)} + \sqrt{(\phi_2-\phi_3)}\mathbf{J}^{(3)}, \\ \sqrt{(\phi_1-\phi_3)}\mathbf{M} &= \sqrt{(\phi_1-\phi_2)}\mathbf{J}^{(1)} - \sqrt{(\phi_2-\phi_3)}\mathbf{J}^{(3)}. \end{aligned} \tag{4}$$

Associated with $\boldsymbol{\phi}$ is the ellipsoid ε: $\phi_{AB}X_AX_B = 1$. This has two central circular sections, each of radius $(\phi_2)^{-1/2}$. They lie on the planes Σ_1 and Σ_2 (say), with unit normals $\mathbf{L}$ and $\mathbf{M}$, respectively.

If $\mathbf{N}$ is any unit vector in the plane Σ_1, then $\mathbf{N}\cdot\mathbf{L} = 0$, and hence using (3):

$$\mathbf{R}_{(\mathbf{N})}\cdot\mathbf{R}_{(\mathbf{N})} = \psi_{iA}N_A\psi_{iB}N_B = \phi_{AB}N_AN_B = \phi_2. \tag{5}$$

Thus, for any $\mathbf{N}$ along a radius of the central circular section in the plane Σ_1, the corresponding $\mathbf{R}_{(\mathbf{N})}$ are each of length $(\phi_2)^{1/2}$.

If $\mathbf{N}$ and $\mathbf{N}'$ are any two unit vectors in the plane Σ_1, and if θ is the angle between them, then using (1) and (3):

$$\mathbf{R}_{(\mathbf{N})}\cdot\mathbf{R}_{(\mathbf{N}')} = \psi_{iA}N_A\psi_{iB}N'_B = \phi_{AB}N_AN'_B = \phi_2N_AN'_A = \phi_2\cos\theta, \tag{6}$$

and hence, using (5), θ is also the angle between $\mathbf{R}_{(\mathbf{N})}$ and $\mathbf{R}_{(\mathbf{N}')}$. Further, using a result on second-order tensors (Chadwick, 1976, p. 20):

$$\begin{aligned} \mathbf{R}_{(\mathbf{N})} \wedge \mathbf{R}_{(\mathbf{N}')} &= (\boldsymbol{\psi}\mathbf{N}) \wedge (\boldsymbol{\psi}\mathbf{N}') = (\det\boldsymbol{\psi})(\boldsymbol{\psi}^{-1})^T(\mathbf{N}\wedge\mathbf{N}') \\ &= (\det\boldsymbol{\psi})\sin\theta\{(\boldsymbol{\psi}^{-1})^T\mathbf{L}\}, \end{aligned} \tag{7}$$

so that for all $\mathbf{N}$ and $\mathbf{N}'$ lying in the plane Σ_1 with unit normal $\mathbf{L}$, the corresponding $\mathbf{R}_{(\mathbf{N})}$ and $\mathbf{R}_{(\mathbf{N}')}$ lie in the plane σ_1 (say) with normal along $(\boldsymbol{\psi}^{-1})^T\mathbf{L}$.

There are thus two circles of special directions if $\phi_1 > \phi_2 > \phi_3$. It may be shown that there are only two (Hayes, 1990).

For the ellipsoid ε the planes of the central circular sections are

$$\phi_{AB}X_AX_B = \phi_2X_AX_A. \tag{8}$$

The connections between the various central circular sections are best visualised by considering the homogeneous deformation

$$\mathbf{x} = \psi\mathbf{X}, \quad \mathbf{X} = \psi^{-1}\mathbf{x}, \tag{9}$$

where ψ is a constant tensor. Then the unit sphere $\mathbf{X}^T\mathbf{X} = 1$ becomes the ellipsoid $\mathbf{x}^T\psi^{-1T}\psi^{-1}\mathbf{x} = 1$ whilst the ellipsoid ε is deformed into the unit sphere $\mathbf{x}^T\mathbf{x} = 1$. We note the identity

$$\begin{aligned}\mathbf{X}^T\psi^T\psi\mathbf{X} - \phi_2\mathbf{X}^T\mathbf{X} &= \mathbf{x}^T\mathbf{x} - \phi_2\mathbf{x}^T\psi^{-1T}\psi^{-1}\mathbf{x} \\ &= -\phi_2\{\mathbf{x}^T\psi^{-1T}\psi^{-1}\mathbf{x} - \phi_2^{-1}\mathbf{x}^T\mathbf{x}\},\end{aligned} \tag{10}$$

so that the central circular sections of radius $(\phi_2)^{1/2}$ of the "material" ellipsoid $\mathbf{X}^T\psi^T\psi\mathbf{X} = 1$ go into the central circular sections of the "spatial" ellipsoid $\mathbf{x}^T\psi^{-1T}\psi^{-1}\mathbf{x} = 1$ of radius $(\phi_2)^{-1/2}$.

Remark. ψ singular. For the purposes of the theorem it is assumed that ψ is not singular. Here we relax this and assume that one of the eigenvalues of ψ is zero. Let the corresponding eigenvector be $\mathbf{S}$ (say). Then because $\psi\mathbf{S} = 0$ it follows also that $\phi\mathbf{S} = \psi^T\psi\mathbf{S} = 0$ and hence we take $\mathbf{S} = \mathbf{J}^{(3)}$. Now ϕ is positive semi-definite. We still have (3) with $\phi_3 = 0$.

In order to associate an ellipsoid with ϕ in this case, we consider $\mathbf{\Phi} = \phi + \beta\mathbf{1}$, where β is positive. The eigenvalues of $\mathbf{\Phi}$, namely $\phi_1 + \beta$, $\phi_2 + \beta$, β are all positive and the corresponding eigenvectors are $\mathbf{J}^{(\alpha)}$. We have

$$\Phi_{AB} = (\phi_2 + \beta)\delta_{AB} + \tfrac{1}{2}\phi_1(H_A^+ H_B^- + H_A^- H_B^+), \tag{11}$$

where

$$(\phi_1)^{1/2}\mathbf{H}^{\pm} = (\phi_1 - \phi_2)^{1/2}\mathbf{J}^{(1)} \pm (\phi_2)^{1/2}\mathbf{J}^{(3)}. \tag{12}$$

Associated with $\mathbf{\Phi}$ is the ellipsoid E: $\Phi_{AB}X_AX_B = 1$. Its two central circular sections are each of radius $(\phi_2 + \beta)^{-1/2}$ and lie on planes $\Sigma^{\pm}$ (say) with normals $\mathbf{H}^{\pm}$.

If $\mathbf{N}$ is any unit vector in the plane Σ^+, then $\mathbf{H}^+ \cdot \mathbf{N} = 0$ and we may write any such $\mathbf{N}$ in the form

$$\mathbf{N} = \mathbf{J}^{(2)}\cos\theta + (\phi_1)^{-1/2}\{(\phi_2)^{1/2}\mathbf{J}^{(1)} - (\phi_1 - \phi_2)^{1/2}\mathbf{J}^{(3)}\}\sin\theta. \tag{13}$$

We have

$$\psi\mathbf{N} = \psi\mathbf{J}^{(2)}\cos\theta + (\phi_2/\phi_1)^{1/2}\psi\mathbf{J}^{(1)}\sin\theta. \tag{14}$$

Now $\mathbf{J}^{(2)}$ and $\mathbf{J}^{(1)}$ are eigenvectors of the real symmetric tensor Φ corresponding to distinct eigenvalues and hence

$$\mathbf{J}^{(1)T}\psi^T\psi\mathbf{J}^{(2)} = \mathbf{J}^{(1)T}\phi J^{(2)} = 0, \tag{15}$$

so that $\psi\mathbf{J}^{(1)}$ and $\psi\mathbf{J}^{(2)}$ are orthogonal. Then directly from (14), as before, we have $\mathbf{R}_{(N)} \cdot \mathbf{R}_{(N)} = \phi_2$, $\mathbf{R}_{(N)} \cdot \mathbf{R}_{(N')} = \phi_2\mathbf{N} \cdot \mathbf{N}'$. Thus for any $\mathbf{N}$ lying along a radius of the central circular section of the Φ ellipsoid in the plane Σ^+, the corresponding $\mathbf{R}_{(N)}$ are each of length $(\phi_2)^{1/2}$, and for $\mathbf{N}$ and $\mathbf{N}'$ lying in Σ^+ the angle between them is equal to the angle between $\mathbf{R}_{(N)}$ and $\mathbf{R}_{(N')}$. (Similar statements are valid for $\mathbf{N}$ lying in Σ^-.)

We may no longer use (7) because ψ^{-1} is not defined. Even so it is clear from (14) that all the vectors $\psi\mathbf{N}$ lie in the plane spanned by the non-zero orthogonal vectors $\psi\mathbf{J}^{(1)}$ and $\psi\mathbf{J}^{(2)}$.

Finally we note that if $\phi_1 = \phi_2 \neq 0$, $\phi_3 = 0$, and if $\mathbf{J}^{(3)}$ is the eigenvector corresponding to the zero eigenvalue then we have

$$\phi_{ij} = \phi_1(\delta_{ij} - J_i^{(3)}J_j^{(3)}). \tag{16}$$

For any $\mathbf{N}$, $\mathbf{N}'$ orthogonal to $\mathbf{J}^{(3)}$, then $\mathbf{R}_{(\mathbf{N})}\cdot\mathbf{R}_{(\mathbf{N})} = \mathbf{N}^T\phi\mathbf{N} = \phi_1$ and $\mathbf{R}_{(\mathbf{N})}\cdot\mathbf{R}_{(\mathbf{N}')} = \phi_1\mathbf{N}\cdot\mathbf{N}'$. Thus in this case there is only one plane of special directions.

Thus the theorem has been extended to include singular tensors with one zero eigenvalue.

Now we turn to some applications.

3. FINITE STRAIN

For the deformation $\mathbf{x} = \mathbf{X}(\mathbf{X})$ taking the particle $\mathbf{X}$ to the place $\mathbf{x}$, let the deformation gradient be denoted by $\mathbf{F}$. Then the material element $\mathrm{d}\mathbf{X}$ at $\mathbf{X}$ is deformed into $\mathrm{d}\mathbf{x}$ at $\mathbf{x}$ where

$$\mathrm{d}\mathbf{x} = \mathbf{F}\,\mathrm{d}\mathbf{X}, \quad \mathrm{d}x_i = F_{iA}\,\mathrm{d}X_A, \tag{17}$$

and

$$F_{iA} = x_{i,A}(\equiv \partial x_i/\partial X_A). \tag{18}$$

Now $\mathbf{F}$ is non-singular and

$$\mathrm{d}\mathbf{X} = \mathbf{F}^{-1}\,\mathrm{d}\mathbf{x}, \quad \mathrm{d}X_A = F_{Ai}^{-1}\,\mathrm{d}x_i, \quad F_{Ai}^{-1} = X_{A,i}(\equiv \partial X_A/\partial x_i). \tag{19}$$

The left and right Cauchy–Green strain tensors $\mathbf{B}$ and $\mathbf{C}$ are given by

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T, \quad \mathbf{C} = \mathbf{F}^T\mathbf{F}, \quad B_{ij} = x_{i,A}x_{j,A}, \quad C_{AB} = x_{i,A}x_{i,B}. \tag{20}$$

The principal stretches at $\mathbf{X}$ are λ_1, λ_2, λ_3. They are the positive square roots of the eigenvalues of $\mathbf{C}$. We assume the ordering $\lambda_1 > \lambda_2 > \lambda_3$.

If $\mathbf{N}$ and $\mathbf{M}$ are unit vectors along material elements at $\mathbf{X}$ and θ is the angle between them, then the shear $\gamma_{(\mathbf{N};\mathbf{M})}$ of the pair of material elements is given by

$$\cos(\theta + \gamma_{(\mathbf{N};\mathbf{M})}) = C_{AB}N_AM_B/[\lambda_{(\mathbf{N})}\lambda_{(\mathbf{M})}], \tag{21}$$

where $\lambda_{(\mathbf{N})}$, the stretch of the element along $\mathbf{N}$, is given by

$$\lambda_{(\mathbf{N})} = \sqrt{C_{AB}N_AN_B}. \tag{22}$$

Let $\mathbf{I}^{(\alpha)}$ be the mutually orthogonal unit eigenvectors of $\mathbf{C}$ with corresponding eigenvalues λ_α^2. Then, as in Section 2, we write

$$C_{AB} = \lambda_2^2\delta_{AB} + \tfrac{1}{2}(\lambda_1^2 - \lambda_3^2)\{H_A^+H_B^- + H_A^-H_B^+\}, \tag{23}$$

where

$$\sqrt{\lambda_1^2 - \lambda_3^2}\,\mathbf{H}^{\pm} = \sqrt{\lambda_1^2 - \lambda_2^2}\,\mathbf{I}^{(1)} \pm \sqrt{\lambda_2^2 - \lambda_3^2}\,\mathbf{I}^{(3)}. \tag{24}$$

The central circular sections of the ellipsoid

$$C_{AB}\,\mathrm{d}X_A\,\mathrm{d}X_B = \alpha^2, \quad |\alpha| \ll 1 \tag{25}$$

lie on the planes

$$C_{AB}\,dX_A\,dX_B = \lambda_2^2\,dX_A\,dX_A. \tag{26}$$

Proceeding as in Section 2 we have the identity

$$(C_{AB}-\lambda_2^2\delta_{AB})\,dX_A\,dX_B = -\lambda_2^2(B_{ij}^{-1}-\lambda_2^{-2}\delta_{ij})\,dx_i\,dx_j \tag{27}$$

so that the central circular sections of the ellipsoid (25) are deformed into the central circular sections of the ellipsoid

$$B_{ij}^{-1}\,dx_i\,dx_j = \alpha^2. \tag{28}$$

All material elements in the planes (26) are stretched by the same amount λ_2. This was the observation of Thomson and Tait (1962, Section 167). However, elements in these planes have the further property (Hayes, 1988)—noted by Thomson and Tait in the case of pure shear—that they suffer no shear. Indeed for a pair of material elements **N** and **M** such that $\mathbf{N}\cdot\mathbf{M} = \cos\theta$, and which lie in the plane whose normal is along $\mathbf{H}^+$ or $\mathbf{H}^-$ it is seen immediately from (23) that

$$C_{AB}N_AN_B = C_{AB}M_AM_B = \lambda_2^2,\quad C_{AB}N_AM_B = \lambda_2^2N_AM_A = \lambda_2^2\cos\theta,$$

and hence from (21), $\cos(\theta+\gamma) = \cos\theta$. Because this is valid for all θ it follows that the shear $\gamma = 0$.

Area elements

We recall that the material vector area element $\Delta\mathbf{A}$ at **X** is deformed into the element $\Delta\mathbf{a}$ at **x** where, by Nanson's formula (Truesdell and Toupin, 1960):

$$\Delta a_m = \det(\mathbf{F})\partial X_K/\partial x_m \Delta A_K. \tag{29}$$

Let

$$\Delta A_K = \Delta A N_K,\quad \Delta a_i = \Delta a n_i. \tag{30}$$

Then

$$(\Delta a)^2 = (\det F)^2 C_{AB}^{-1} N_A N_B (\Delta A)^2. \tag{31}$$

For **N** lying along radii of a central circular section of the ellipsoid of $\mathbf{C}^{-1}$: $C_{AB}^{-1}X_AX_B = 1$, we have

$$(\Delta a)^2 = (\det \mathbf{F})^2\lambda_2^{-2}(\Delta A)^2 = \lambda_1^2\lambda_3^2(\Delta A)^2, \tag{32}$$

and thus

$$\Delta a = \lambda_1\lambda_3\Delta A. \tag{33}$$

Hence all material areal elements with normals **N** along radii of a central circular section of the ellipsoid of $\mathbf{C}^{-1}$ are subjected to the areal magnification $\lambda_1\lambda_3$. Further, the angle between the normals **N** is preserved. Thus if $\mathbf{N}'$ and $\mathbf{N}''$ are two such normals in the undeformed body then the corresponding equal areal elements $\Delta A\mathbf{N}'$ and $\Delta A\mathbf{N}''$ are deformed into two equal areal elements $\Delta\mathbf{a}' = \lambda_1\lambda_3\Delta A\mathbf{n}'$ and $\Delta\mathbf{a}'' = \lambda_1\lambda_3\Delta A\mathbf{n}''$ where $\mathbf{n}'\cdot\mathbf{n}'' = \mathbf{N}'\cdot\mathbf{N}''$.

4. INFINITESIMAL STRAIN

Let

$$\mathbf{x} = \mathbf{X} + \mathbf{U}(\mathbf{X}), \quad x_i = X_i + U_i(\mathbf{X}) \tag{34}$$

where $\mathbf{U}$ is the displacement. Then within the context of the infinitesimal strain theory, $\mathbf{C}$ is approximated by

$$C_{AB} = \delta_{AB} + 2E_{AB}, \quad 2E_{AB} = \frac{\partial U_A}{\partial X_B} + \frac{\partial U_B}{\partial X_A}. \tag{35}$$

The strain tensor $\mathbf{E}$ is not in general positive definite. Indeed in the very basic case of simple shear $\mathbf{E}$ is singular. We may however associate with it an ellipsoid $(E_{AB} + \beta\delta_{AB})X_A X_B = 1$ where β is a suitably large positive scalar. The eigenvalues of $\mathbf{E} + \beta\mathbf{1}$ are $(E_\alpha + \beta)$ where E_α are the eigenvalues of $\mathbf{E}$.

We assume $E_1 > E_2 > E_3$. The stretch of a material element along $\mathbf{N}$ at $\mathbf{X}$ is $1 + E_{AB}N_A N_B$.

The increase γ in the angle θ between the pair of material elements along $\mathbf{N}$ and $\mathbf{M}$ at $\mathbf{X}$ is given by Thomas (1961):

$$(-\sin\theta)\gamma = 2E_{AB}N_A M_B - E_{PQ}(N_P N_Q + M_P M_Q)\cos\theta. \tag{36}$$

We note that if in (36), $\mathbf{E}$ is replaced by $\mathbf{E} + \beta\mathbf{1}$ then γ remains unchanged. This is because the replacement of $\mathbf{E}$ by $\mathbf{E} + \beta\mathbf{1}$ is equivalent to the superposition upon the given displacement field $\mathbf{U}$ of the further radial displacement $\mathbf{U} = \beta\mathbf{X}$ which should not affect the shear of elements. Such superposition is additive in the case of infinitesimal strain.

For material elements lying in the plane of the central circular section of the ellipsoid $(E_{AB} + \beta\delta_{AB})X_A X_B = 1$ at $\mathbf{X}$, we find, as before, that these elements are all stretched by the same amount $(1 + E_2)$, and further, pairs of such elements are not sheared. If $\mathbf{K}^{(\alpha)}$ are the unit eigenvectors of $\mathbf{E}$, corresponding to the eigenvalues E_α, then $\mathbf{K}^{(\alpha)}$ are also the eigenvectors of $\mathbf{E} + \beta\mathbf{1}$, and the normals to the planes of the central circular sections are along $\mathbf{K}^{\pm}$ (say) given by

$$\mathbf{K}^{\pm} = (E_1 - E_2)^{1/2}\mathbf{K}^{(1)} \pm (E_2 - E_3)^{1/2}\mathbf{K}^{(3)}. \tag{37}$$

Area elements

Within the context of infinitesimal strain

$$\frac{\partial X_i}{\partial x_j} = \delta_{ij} - \frac{\partial U_i}{\partial X_j}, \quad C_{AB}^{-1} = \delta_{AB} - 2E_{AB}, \quad \det(F) = 1 + E_{AA}, \tag{38}$$

and equations (29) and (31) become

$$\Delta a_m = (1 + E_{AA})(\delta_{Km} - \partial U_K/\partial X_m)\Delta A_K,$$
$$\Delta a = \{1 + E_{AA} - E_{KM}N_K N_M\}\Delta A, \tag{39}$$

on using (30). Thus the ratio of the change in area per unit initial area for a material areal element whose normal is along $\mathbf{N}$ is

$$(\Delta a - \Delta A)/\Delta A = (\delta_{KM} - N_K N_M)E_{KM}. \tag{40}$$

If $\mathbf{N}$ is orthogonal to $\mathbf{K}^+$ or $\mathbf{K}^-$, then $E_{KM}N_K N_M = E_2$, and $(\Delta a - \Delta A)/\Delta A = E_1 + E_3$.

Thus all material areal elements with normals **N** along radii of a central circular section of the ellipsoid of $\mathbf{E}+\beta\mathbf{1}$ are subjected to the area magnification $1+E_1+E_3$. Further, the angle between the normals is preserved.

5. RATE OF STRAIN

From (17), using a dot ($\cdot$) to denote the material time derivative we have

$$\dot{\overline{\mathrm{d}x_i}} = (\partial v_i/\partial x_j)\,\mathrm{d}x_j. \tag{41}$$

For the material element $\mathrm{d}\mathbf{x} = \mathrm{d}x\mathbf{n}$ at **x**, it may be shown (Truesdell and Toupin, 1960) that

$$\dot{\overline{\mathrm{d}x}} = \mathrm{d}_{ij}n_in_j\,\mathrm{d}x, \tag{42}$$

where d_{ij} is the rate of strain tensor

$$2\mathrm{d}_{ij} = (\partial v_i/\partial x_j + \partial v_j/\partial x_i). \tag{43}$$

If the angle between two material elements $\mathrm{d}\mathbf{x}^{(1)}$ and $\mathrm{d}\mathbf{x}^{(2)}$ at **x** be denoted by α, and if $\mathrm{d}\mathbf{x}^{(1)} = \mathrm{d}x^{(1)}\mathbf{n}$, $\mathrm{d}\mathbf{x}^{(2)} = \mathrm{d}x^{(2)}\mathbf{m}$ then

$$\mathrm{d}x_i^{(1)}\,\mathrm{d}x_i^{(2)} = \mathrm{d}x^{(1)}\,\mathrm{d}x^{(2)}\cos\alpha. \tag{44}$$

It may be shown that the shearing, $\dot{\alpha}$, of the pair of material elements instantaneously directed along **n** and **m** at **x**, is given by

$$-(\sin\alpha)\dot{\alpha} = 2\mathrm{d}_{ij}n_im_j - \mathrm{d}_{ij}(n_in_j+m_im_j)\cos\alpha. \tag{45}$$

We note that if the motion is changed from v_i to v_i+kx_i where k is a constant, then d_{ij} is changed to $\mathrm{d}_{ij}+k\delta_{ij}$, $\mathrm{d}_{ij}n_im_j$ becomes $\mathrm{d}_{ij}n_im_j+k\cos\alpha$, $\mathrm{d}_{ij}n_in_j$ becomes $\mathrm{d}_{ij}n_in_j+k$. But the expression (45) is unchanged. Thus we associate with the motion the "**d**" ellipsoid $(\mathrm{d}_{ij}+k\delta_{ij})x_ix_j = 1$ where k is a suitably large constant.

For material elements instantaneously lying in a plane of a central circular section of the **d** ellipsoid, the stretchings are all equal to d_2 where d_α (with $d_1 > d_2 > d_3$) are the principal stretchings, the eigenvalues of d_{ij}. These eigenvalues may take any values.

Also if **n** and **m** are directed along material elements lying instantaneously in the plane of the central circular section of the **d** ellipsoid, then $\mathrm{d}_{ij}n_im_j = \mathrm{d}_2\cos\alpha$, $\mathrm{d}_{ij}n_in_j = \mathrm{d}_{ij}m_im_j = \mathrm{d}_2$ and hence $\dot{\alpha} = 0$. Thus there is no shearing for material elements lying instantaneously in a plane of a central circular section of the **d** ellipsoid. The normals to these planes are $\mathbf{S}^{\pm}$ (say) given by

$$\mathbf{S}^{\pm} = \{(\mathrm{d}_1-\mathrm{d}_2)^{1/2}\mathbf{e}_1 \pm (\mathrm{d}_2-\mathrm{d}_3)^{1/2}\mathbf{e}_3\}(\mathrm{d}_1-\mathrm{d}_3)^{-1/2}, \tag{46}$$

where $\mathbf{e}_\alpha$ are the unit eigenvectors of **d** corresponding to the eigenvalues d_α ($\alpha = 1, 2, 3$).

6. STRESS

The traction $\mathbf{T}_{(\mathbf{N})}$ across a material element with normal **N** at **X** in the reference configuration is given by

$$\mathbf{T}_{(\mathbf{N})} = \Pi\mathbf{N}, \quad T_{(\mathbf{N})i} = \Pi_{iA}N_A, \tag{47}$$

where Π is the Piola–Kirchhoff stress tensor. If Π is known to have at most one zero eigenvalue, then $|\mathbf{T}_{(\mathbf{N})}|$ has the same value for all $\mathbf{N}$ lying in the plane(s) g (say), the central circular section(s) of the ellipsoid $\mathbf{X}^T(\Pi^T\Pi+\beta\mathbf{1})\mathbf{X} = 1$, where β is a positive constant. Further, for any pair $\mathbf{N}'$ and $\mathbf{N}''$ lying in g, the angle between $\mathbf{T}_{(\mathbf{N}')}$ and $\mathbf{T}_{(\mathbf{N}'')}$ is equal to the angle between $\mathbf{N}'$ and $\mathbf{N}''$.

The traction $\mathbf{t}_{(\mathbf{n})}$ across a material element with normal $\mathbf{n}$ at $\mathbf{x}$ in the current configuration is given by

$$\mathbf{t}_{(\mathbf{n})} = \mathbf{t}\mathbf{n}, \quad t_{(\mathbf{n})i} = t_{ij}n_j, \tag{48}$$

where t_{ij} is the (symmetric) Cauchy stress tensor. Again, if $\mathbf{t}$ is known to have at most one zero eigenvalue, then $|\mathbf{t}(\mathbf{n})|$ has the same value for all $\mathbf{n}$ lying in the plane(s) σ (say) of the central circular section(s) of the ellipsoid $\mathbf{x}^T(\mathbf{t}^T\mathbf{t}+\beta\mathbf{1})\mathbf{x} = 1$ where β is a positive constant. Further, for any pair $\mathbf{n}'$ and $\mathbf{n}''$ lying in σ, the angle between $\mathbf{t}_{(\mathbf{n}')}$ and $\mathbf{t}_{(\mathbf{n}'')}$ is equal to the angle between $\mathbf{n}'$ and $\mathbf{n}''$.

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